MAT 1322
Lecture 21

03/04/201

APPLICATION OF THE TOTAL DIFFERENTIAL

recall: $f(x,y,z)$'s total differential:

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial z} dz$$

→ use it to estimate error (similar to examples @ beginning of course)

__Ex__ (pg 933)

$h = 25\text{cm}$
$r = 10\text{cm}$

But! There's a possible error in measurement of 0.1 cm (= 1mm)

Based on this what is the possible error that we make if we compute the volume based on these measurements?

volume formula: $V = r^2 \pi h$

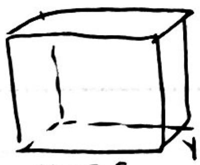
We can compute the error with the total differential:

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = 2r\pi \, dr + r^2 \pi \, dh$$

To maximize the error, we replace dr and dh by 0.1cm:

$$dV = 2r\pi h \cdot 0.1 + r^2 \pi \cdot 0.1 = 500\pi \cdot 0.1 + 100\pi \cdot 0.1$$
$$= 600\pi (0.1) = 60\pi$$

__Ex 2__ Box:



$z = 40$
$y = 60$
$x = 75$

error in measurement: 0.2

$V = xyz$

total differential: $dV = \underset{\frac{\partial V}{\partial x}}{\underset{\parallel}{yz}}\, dx + \underset{\frac{\partial V}{\partial y}}{\underset{\parallel}{xz}}\, dy + \underset{\frac{\partial V}{\partial z}}{\underset{\parallel}{xy}}\, dz$

$$dV = 60 \cdot 40 \cdot 0.2 + 75 \cdot 40 \cdot 0.2 + 75 \cdot 60 \cdot 0.2 = 1980 \text{ cm}^3$$

Extra info about the gradient vector:

recall gradient vector (x,y,z)

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$$

relation between ∇f and directional derivative $df\,\vec{u}$: the directional derivative is maximized. if $\vec{u} = \nabla f$

maximum increase: in direction of ∇f

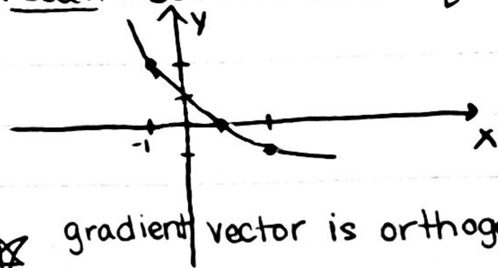
Relation to level curves: Ex $z = f(x,y) = x^2y + xz$

Set $z=1$: $1 = x^2y + x$

$z=2$: $2 = x^2y + 2x$

$z=-1$: $-1 = x^2y - x$

recall: curves are equation



values $z=1$

x	y	
-1	2	$\leftarrow -1 = y - 1$
1	0	$\leftarrow 1 = y + 2$
2	$\frac{-1}{4}$	$1 = 2^2y + 2 \Leftrightarrow -1 = 4y$

✯ gradient vector is orthogonal to the level curves

or: express y explicitly; set $z=k$ (k was 1,2,-2)

$$k = x^2y + x \cdot k$$

$$k - xk = x^2y \qquad /: x^2 \ (x=0 \text{ not defined})$$

$$\underline{\underline{y}} = \frac{k-xk}{x^2} = \frac{k}{x^2} - \frac{k}{x} = \underline{\underline{k\left(\frac{1}{x^2} - \frac{1}{x}\right)}}$$

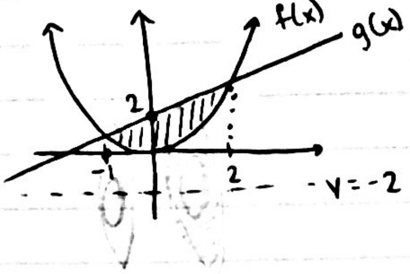
from this we get the level curve equations by setting $k = -2, -1, 0, 1, 2$.

area under curves likely not on exam ("too easy")

REVIEW

Volumes of rotational solids

Ex $f(x) = x^2$, $g(x) = x+2$, rotate around $y = -2$.



① find POIs

$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\boxed{x=2}, \boxed{x=-1}$

cross-section area: annulus

② if we rotate around y=k, integrate over x.

→ cross section area formula:

③ $A(x) = (r_o^2 - r_i^2)\pi$

$= \left((g(x) - (-2))^2 - (f(x)+2)^2\right)\pi$

$= \left((x+2+2)^2 - (x^2+2)^2\right)\pi$

$= (x^2 + 8x + 16 - x^4 - 4x^2 - 4)\pi$

$= (-x^4 - 3x^2 + 8x + 12)\pi$ → this is ≥ 0 between $x=-1$ & $x=2$

④ $V = \pi \int_{-1}^{2} (-x^4 - 3x^2 + 8x + 12)\,dx$

$= \pi \left[\frac{-x^5}{5} - \frac{3x^3}{3} + \frac{8x^2}{2} + 12x \right]_{-1}^{2}$

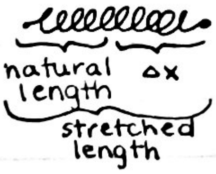
$= \pi \left[-\frac{x^5}{5} - x^3 + 4x^2 + 12x \right]_{-1}^{2}$

$= \pi \left[\frac{-32}{5} - 8 + 16 + 24 - \left(-\frac{(-1)^5}{5} - (-1)^3 + 4(-1)^2 + 12(-1)\right) \right]$

$\boxed{V = \frac{16}{5}\pi}$

Work

Springs



Hooke's Law: $F = k\Delta x$ ~k is const. depending on spring

(force needed to keep it stretched → F)

Ex Given: need 40N to extend to 30cm.

natural length: 20 cm ; $\Delta x = 10$cm ; $W = F \times d$

$F = k\Delta x$ → $\underline{F = 400\Delta x}$

$40 = k(0.1)$

$k = 400\ \text{N/m}$

ALWAYS NEED METRES

QUESTION: W = ? to stretch 10 more cm:

0.3 → 0.4 m

*here: work is always integration over force w.r.t. d

$F(x) = k\Delta x = k(x - x_o)$, x_o - natural length

force function; varies with x

⟶

$$W = \int_{0.3m}^{0.4m} 400(x-0.2)\,dx$$

$$= 400\int_{0.3}^{0.4}(x-0.2)\,dx$$

$$= 400\left(\frac{x^2}{2}-0.2x\right)_{0.3}^{0.4}$$

$$= 400\left(\frac{0.4^2}{2}-0.2(0.4)-\left(\frac{0.3^2}{2}-0.2(0.3)\right)\right)$$

$$= 6\text{ J}$$